2-1. Unitary operator

$$\left.\hat{T}(dx'),\ \hat{u}(t,t_0)\right\} \longrightarrow \hat{U}$$

if $|\alpha'\rangle = \hat{U}|\alpha\rangle \qquad |\beta'\rangle = \hat{U}|\beta\rangle$

$$\Rightarrow \langle\beta'|\alpha'\rangle = \langle\hat{U}\beta|(\hat{U}\alpha)\rangle = \langle\beta|\underbrace{\hat{U}^{+}\hat{U}}_{\Downarrow\ \mathbb{1}}|\alpha\rangle = \langle\beta|\alpha\rangle$$

Now $\langle\beta'|\hat{x}|\alpha'\rangle$

$= \langle\hat{U}\beta|\hat{x}|(\hat{U}\alpha)\rangle$

1> $= (\langle\beta|\hat{U}^{+})\,\hat{x}\,(\hat{U}|\alpha\rangle)$ → Schrödinger " $\hat{x}^{(S)}$ "

2> $= \langle\beta|\underline{(\hat{U}^{+}\hat{x}\hat{U})}|\alpha\rangle$ → Heisenberg. " $\hat{x}^{(H)}$ "

→ New observable (changes)

e.g. $\hat{U} = \hat{T}(dx')$, $|\alpha'\rangle = \hat{T}(dx')|\alpha\rangle$

1> if $\hat{x}|\alpha\rangle = x'|\alpha\rangle \Rightarrow \langle\hat{x}\rangle_{\alpha'} = x' + dx'$

2> if keeping $|\alpha\rangle$

$$\underset{\Downarrow\ \hat{x}^{(H)}}{\hat{x}'} = \hat{U}^{+}\underset{\Downarrow\ \hat{x}^{(S)}}{\hat{x}}\hat{U} = \hat{T}^{+}\hat{x}\hat{T}$$

$$= (\mathbb{1} + \tfrac{i}{\hbar}\hat{p}^{+}dx')\,\hat{x}\,(\mathbb{1} - \tfrac{i}{\hbar}\hat{p}\,dx')$$

$$= (\mathbb{1} + \tfrac{i}{\hbar}\hat{p}^{+}dx')(\hat{x} - \tfrac{i}{\hbar}\hat{x}\hat{p}\,dx')$$

$$= \hat{x} - \tfrac{i}{\hbar}\hat{x}\hat{p}\,dx' + \tfrac{i}{\hbar}\underbrace{\hat{p}^{+}}_{\hookrightarrow\hat{p}}dx'\,\hat{x} + \overset{\text{2-order}}{\mathcal{O}}(dx')$$

$$= \hat{x} - \underbrace{[\hat{x}, \hat{p}]}_{\Downarrow\ i\hbar\delta_{ij}\mathbb{1}} dx' \frac{i}{\hbar} = \hat{x} + dx'$$

$$\Rightarrow \hat{x}^{(H)} = \hat{x}' = \hat{x} + dx'$$

$$\Rightarrow \langle\alpha|\hat{x}'|\alpha\rangle = \langle\alpha|(\hat{x} + dx')|\alpha\rangle$$

$$= \langle\alpha|(x' + dx')|\alpha\rangle = x' + dx'$$

$$\Rightarrow \underset{\substack{\downarrow \\ H}}{\langle\hat{x}'\rangle_\alpha} = \underset{\substack{\downarrow \\ S}}{\langle\hat{x}\rangle_{\alpha'}}$$

2-2. state kets and observables in schrödinger & Heisenberg picture.

1> (S) QM: state varying. operator fixed

2> (H) QM: state fixed, operator varying

Heisenberg observable ($\hat{A}^{(H)}$):

$$\hat{A}^{(H)} = \hat{U}^{+} \hat{A}^{(S)} \hat{U}$$

if dynamics $\hat{U} = \hat{u}(t)$

$$\Rightarrow \hat{A}^{(H)}(t) = \hat{u}^{+}(t) \hat{A}^{(S)} \hat{u}(t)$$

$$\Rightarrow \hat{A}^{(H)}(t=0) = \hat{A}^{(S)}$$

Heisenberg state: $|\alpha, t_0; t\rangle^{(H)} = |\alpha, t_0; t = t_0\rangle$

$$\Rightarrow \; |\alpha\rangle^{(H)} = \hat{u}^{\dagger}(t)\,|\alpha, t_0; t\rangle^{(S)} = \hat{u}^{\dagger}\hat{u}\,\underline{|\alpha; t_0\rangle}$$

$$= \hat{u}^{-1}(t)\,|\alpha, t_0; t\rangle^{(S)} = \hat{u}^{\dagger}\,|\alpha, t_0; t\rangle$$

$$\langle \hat{A} \rangle_{\alpha} = \underline{{}^{(S)}\langle \alpha, t_0; t|\, \hat{A}^{(S)}\, |\alpha, t_0; t\rangle^{(S)}}$$

$$= \underbrace{{}^{(S)}\langle \alpha, t_0; t|\, \hat{u}}_{\Downarrow \; \langle\alpha|^{(H)}} \underbrace{\hat{u}^{\dagger}\, \hat{A}^{(S)}\, \hat{u}}_{\Downarrow \; \hat{A}^{(H)}} \underbrace{\hat{u}^{\dagger}\, |\alpha, t_0; t\rangle^{(S)}}_{\Downarrow \; |\alpha\rangle^{(H)}}$$

$$\parallel$$

$$= \underline{\langle\alpha|^{(H)}\, \hat{A}^{(H)}\, |\alpha\rangle^{(H)}} \qquad \hat{A}^{(H)} \rightarrow \hat{A}^{(H)}(t)$$

2-3 Heisenberg Equation of Motion:

$$\hat{A}^{(H)}(t) = \hat{u}^{\dagger}\, \underbrace{\hat{A}^{(S)}}_{\downarrow}\, \hat{u}$$

t-dependent OR t-independent.

$$\frac{d}{dt}\hat{A}^{(H)}(t) = \underbrace{\frac{d\hat{u}^{\dagger}}{dt}}_{\downarrow \; \frac{i}{\hbar}\hat{u}^{\dagger}\hat{H}^{(S)}} \hat{A}^{(S)}\hat{u} + \hat{u}^{\dagger}\frac{d\hat{A}^{(S)}}{dt}\hat{u} + \hat{u}^{\dagger}\hat{A}^{(S)} \underbrace{\frac{d\hat{u}}{dt}}_{\Downarrow \; -\frac{i}{\hbar}\hat{H}^{(S)}\hat{u}}$$

$$= \frac{i}{\hbar}\hat{u}^{\dagger}\hat{H}^{(S)}\hat{A}^{(S)}\hat{u} + \hat{u}^{\dagger}\hat{A}^{(S)}\left(-\frac{i}{\hbar}\hat{H}^{(S)}\hat{u}\right) + \hat{u}^{\dagger}\frac{d\hat{A}^{(S)}}{dt}\hat{u}$$

$$= \frac{i}{\hbar}\underbrace{\hat{u}^{\dagger}\hat{H}^{(S)}\hat{u}}_{\hat{H}^{(H)}}\underbrace{\hat{u}^{\dagger}\hat{A}^{(S)}\hat{u}}_{\hat{A}^{(H)}} + \underbrace{\hat{u}^{\dagger}\hat{A}^{(S)}\hat{u}}_{\hat{A}^{(H)}}\underbrace{\hat{u}^{\dagger}\left(-\frac{i}{\hbar}\hat{H}\hat{u}\right)}_{\hat{H}^{(H)}} + \hat{u}^{\dagger}\frac{d\hat{A}^{(S)}}{dt}\hat{u}, \qquad \underbrace{\hat{u}\hat{u}^{\dagger}}_{1}$$

$$= \frac{i}{\hbar} \hat{H}^{(H)} \hat{A}^{(H)} - \frac{i}{\hbar} \hat{A}^{(H)} \hat{H}^{(H)} + \hat{u}^{+} \frac{d\hat{A}^{(S)}}{dt} \hat{u}$$

$$= \frac{i}{\hbar} [\hat{H}^{(H)}, \hat{A}^{(H)}] + \hat{u}^{+} \frac{d\hat{A}^{(S)}}{dt} \hat{u}$$

$$= \frac{1}{i\hbar} [\hat{A}^{(H)}, \hat{H}^{(H)}] + \left[\frac{d\hat{A}^{(S)}}{dt} \right]^{(H)}$$

$$\boxed{\frac{d\hat{A}^{(H)}(t)}{dt} = \frac{1}{i\hbar} [\hat{A}^{(H)}, \hat{H}^{(H)}] + \left[\frac{d\hat{A}^{(S)}}{dt} \right]^{(H)}} \quad *$$

Heisenberg Eq. of Motion. for dynamics

⇒ in (S) t-evolution is captured by state.

i.e. $|\alpha, t_0; t\rangle = \hat{u} |\alpha; t_0\rangle$

Dynamics is given by (S) eq. : observable can t-dependent or t-independent.

⇒ in (H) t-evolution is captured by observables

state is "frozen"

*. even if observables are t-independent in (S) they are in general t-dependent in (H)

For conservative sys. $\Rightarrow$ $\hat{H}^{(S)}(t) = \hat{H}^{(S)}$

(i.e., sys has t-independent $\hat{H}$)

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}^{(S)}t}$$

if $[\hat{X}, \hat{Y}] = 0 \Rightarrow [\hat{X}, \underline{f(\hat{Y})}] = 0$

since $f(\hat{Y})$ can be expanded.

since $[\hat{H}^{(S)}, \hat{H}^{(S)}] = 0 \Rightarrow [\hat{H}^{(S)}, \hat{U}(t)] = 0$

$$\Rightarrow \boxed{\hat{H}^{(H)} = \hat{U}^{+}\hat{H}^{(S)}\hat{U} = \hat{U}^{+}\hat{U}\hat{H}^{(S)} = \hat{H}^{(S)}}$$

$$\Rightarrow \boxed{\frac{d\hat{A}^{(H)}}{dt} = \frac{1}{i\hbar}[\hat{A}^{(H)}, \hat{H}]}$$ *.

if $\hat{H}^{(S)}$ is t-dependent:

case 1> $[\hat{H}^{(S)}(t), \hat{H}^{(S)}(t')] = 0 \quad \forall\, t, t'$

$$\hat{U} = e^{-\frac{i}{\hbar}\int_0^t \hat{H}^{(S)}(t')\,dt'}$$

$$\Rightarrow \hat{H}^{(H)} = \hat{U}^{+}\hat{H}^{(S)}\hat{U} = e^{\frac{i}{\hbar}\int_0^t \hat{H}^{(S)}(t')dt'}\,\hat{H}^{(S)}.\,e^{-\frac{i}{\hbar}\int_0^t \hat{H}^{(S)}(t')dt'}$$

$$= \hat{H}^{(S)}(t)$$

$$\Rightarrow \hat{H}^{(H)}(t) = \hat{H}^{(S)}(t)$$

case 2> if $[\hat{H}^{(S)}(t), \hat{H}^{(S)}(t')] \neq 0$

$$\Rightarrow \hat{H}^{(H)}(t) \neq \hat{H}^{(S)}(t)$$