then $\psi_1(x') = \langle x'|1\rangle = \langle x'|\hat{a}^+|0\rangle$

$$= \frac{1}{\sqrt{2}x_0}\left(x' - x_0^2\frac{d}{dx'}\right)\underbrace{\langle x'|0\rangle}_{\psi_0(x')}$$

$$\psi_2(x') = \langle x'|\hat{a}^+|1\rangle/\sqrt{2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}x_0}\right)^2\left(x' - x_0^2\frac{d}{dx'}\right)^2\langle x'|0\rangle$$

$\vdots$

$$\psi_n(x') = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}x_0}\right)^n\left(x' - x_0^2\frac{d}{dx'}\right)^n\underline{\langle x'|0\rangle}$$

3-2. t-dependent QHO (Heisenberg)

$$\boxed{\frac{d\hat{A}^{(H)}}{dt} = \frac{1}{i\hbar}[\hat{A}^{(H)}, \hat{H}]} \; *$$

$\frac{1}{2}m\omega^2x^2$

$$\Rightarrow \frac{d\hat{P}}{dt} = \frac{1}{i\hbar}[\hat{P}, \hat{H}] = -\frac{\partial V(\hat{x})}{\partial\hat{x}} = -m\omega^2x$$

$$\Rightarrow \frac{d\hat{x}}{dt} = \frac{1}{i\hbar}[\hat{x}, \hat{H}] = \frac{\hat{P}}{m}$$

$$\Rightarrow \frac{d\hat{a}}{dt} = \frac{d}{dt}\left(\sqrt{\frac{m\omega}{2\hbar}}\left[\hat{x} + \frac{i\hat{P}}{m\omega}\right]\right) = -i\omega\hat{a}$$

$$\Rightarrow \frac{d\hat{a}^+}{dt} = i\omega\hat{a}^+$$

$$\Rightarrow \boxed{\hat{a}(t) = \hat{a}(0)e^{-i\omega t}} \quad \boxed{\hat{a}^+ = \hat{a}^+(0)e^{i\omega t}}$$

$$\Rightarrow \hat{N} = \hat{a}^{+}\hat{a} = \hat{a}^{+}(0)\,\hat{a}(0) \Rightarrow \hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$\longrightarrow$ Both $\hat{N}$ & $\hat{H}$ are t-independent

$\hat{x}(t)$, $\hat{p}(t)$

$$\boxed{\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) \\ \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \end{aligned}}$$

$$\Rightarrow \hat{x}(t) + i\frac{\hat{p}(t)}{m\omega} = \hat{x}(0)e^{-i\omega t} + i\frac{\hat{p}(0)}{m\omega}e^{-i\omega t}$$

$$\hat{x}(t) - i\frac{\hat{p}(t)}{m\omega} = \hat{x}(0)e^{i\omega t} - i\frac{\hat{p}(0)}{m\omega}e^{i\omega t}$$

Equate the Hermitian & Non-Hermitian parts.

$$\Rightarrow \hat{x}(t) = \hat{x}(0)\cos\omega t + \left[\frac{\hat{p}(0)}{m\omega}\right]\sin\omega t$$

$$\hat{p}(t) = -m\omega\,\hat{x}(0)\sin\omega t + \hat{p}(0)\cos\omega t$$

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4. Schrödinger wave equation

4-1 t-dependent wave equation

$$\psi_\alpha(x', t) = \langle x' | \underbrace{\alpha, t_0 = 0; t\rangle}_{\Downarrow}$$

Schrödinger picture

$$(|\alpha\rangle \to \{|x'\rangle\})$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \underline{V(\hat{x})}$$

$\longrightarrow$ Hermitian ("Local": t-independent)

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle = \hat{H}|\alpha, t_0; t\rangle$$

$$\Rightarrow \langle x'|\, i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle = \langle x'|\hat{H}|\alpha, t_0; t\rangle$$

$$i\hbar\frac{\partial}{\partial t}\underbrace{\langle x'|\alpha, t_0; t\rangle}_{\psi_\alpha(x', t)} = \sum_{x''}\langle x'|\hat{H}|x''\rangle\langle x''|\alpha, t_0; t\rangle$$

$$= \sum_{x''}\langle x'|\Big(\underline{\frac{\hat{p}^2}{2m}} + \underline{V(\hat{x})}\Big)|x''\rangle\langle x''|\alpha\rangle$$

$$\langle x'|\underbrace{V(\hat{x})}_{\text{Expansion}}|x''\rangle = \langle x'|V(x'')|x''\rangle = V(x'')\langle x'|x''\rangle$$

$$\langle x'|\hat{p}^n|\alpha\rangle = (-i\hbar)^n\frac{\partial^n}{\partial x'^n}\langle x'|\alpha\rangle$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \& \text{ if 3D} \rightarrow \nabla = \Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big)$$

$$\text{if } \hat{p}^n = \hat{p}^2 \Rightarrow \hat{p}^2 = -\hbar^2\nabla^2$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\psi_\alpha(x', t) = \sum_{x''}\underline{\langle x'|}\Big(-\frac{\hbar^2}{2m}\nabla^2 + V(x'')\Big)\underline{|x''\rangle}\langle x''|\alpha\rangle$$

$$\langle x'|x''\rangle = \delta_{x'x''}$$

$$= \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(x')\Big]\underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x', t)}$$

*
$$\boxed{i\hbar\frac{\partial}{\partial t}\psi_\alpha(x', t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(x')\Big]\psi_\alpha(x', t)}$$

Schrödinger wave equation.

4-2. t-independent wave equation

stationary state : $|a'\rangle$ $\left(\begin{array}{l} \hat{H}|a'\rangle = E_{a'}|a'\rangle \\ \hat{A}|a'\rangle = a'|a'\rangle \end{array} \right)$

$\psi_{a'}(x') = \langle x'|a'\rangle$

$\Rightarrow \psi_\alpha(x', t) = \langle x'|\alpha, t_0; t\rangle$ $(|\alpha, t_0; t=t_0\rangle = |a'\rangle)$

$= \langle x'|\hat{u}(t_0, t)|a'\rangle$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\nabla \underbrace{\langle x'|a'\rangle}_{\psi_{a'}(x')} + V(x')\langle x'|a'\rangle = E_{a'}\langle x'|a'\rangle}$$

*. t-independent schrödinger eq.

4-3. Probability current (or flux)

* $\psi_\alpha(x', t) = \langle x'|\alpha, t_0; t\rangle$: Expansion coefficient. of $|\alpha, t_0; t\rangle$ in terms of position eigenket $\{|x'\rangle\}$

*. Def: Probability density

$$\rho(x', t) = |\psi_\alpha(x', t)|^2 = |\langle x'|\alpha, t_0; t\rangle|^2$$

$$\Rightarrow \psi_\alpha^* \psi_\alpha$$

$\Rightarrow$ the probability of recording the presence of a particle in a volume d^3x' is : $\rho(x', t)\, d^3x'$