Time-dependent Expectation Value:

Under (H) picture:

$$\frac{d}{dt}\langle \hat{A} \rangle_\alpha = \frac{d}{dt} \langle \alpha |^{(H)} \underbrace{\hat{A}^{(H)}(t)}_{\hat{U}^{+} \hat{A}^{(S)} \hat{U}} | \alpha \rangle^{(H)}$$

$$= \frac{d}{dt} \langle \alpha |^{(S)} \hat{A}^{(S)} | \alpha \rangle^{(S)}$$

$$= \langle \alpha |^{(H)} \frac{d}{dt} \hat{A}^{(H)}(t) | \alpha \rangle^{(H)}$$

$$= \langle \alpha |^{(H)} \left(\boxed{\frac{d\hat{A}^{(H)}(t)}{dt} = \underbrace{\frac{1}{i\hbar}[\hat{A}^{(H)}, \hat{H}^{(H)}]}_{\hat{X}} + \underbrace{[\frac{d\hat{A}^{(S)}}{dt}]^{(H)}}_{\hat{Y}}} \right) | \alpha \rangle^{(H)}$$

$$= \langle \hat{X} \rangle_\alpha + \langle \hat{Y} \rangle_\alpha$$

$$\boxed{\frac{d}{dt}\langle \hat{A} \rangle_\alpha = \frac{1}{i\hbar} \langle [\hat{A}^{(H)}, \hat{H}^{(H)}] \rangle_\alpha + \langle [\frac{d\hat{A}^{(S)}}{dt}]^{(H)} \rangle_\alpha}$$

The result $\langle \hat{A}^{(H)}(t) \rangle = \langle \hat{A}^{(S)}(t) \rangle$

even if $t \neq t'$ & $[\hat{A}^{(S)}(t), \hat{A}^{(S)}(t')] \neq 0$

2-4. Ehrenfest's Theorem

Describes how expectation values of QM operators evolve in time & establish connection between QM and classic mechanics.

if $\hat{A}^{(S)}$ is t-independent

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{A}\rangle_\alpha = \frac{1}{i\hbar}\langle[\hat{A}^{(H)}, \hat{H}]\rangle_\alpha}$$

For a free particle: no $V(\hat{x})$, only $\hat{P}$

$$\Rightarrow \quad \hat{H} = \frac{\hat{P}^2}{2m} = \frac{\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2}{2m}$$

since $[\hat{P}_i^{(H)}(t), \hat{H}^{(H)}] = 0$

$$\Rightarrow \quad \frac{d}{dt}\underbrace{\hat{P}_i^{(H)}(t)}_{\sim\hat{A}^{(H)}} = \frac{1}{i\hbar}[\hat{P}_i^{(H)}, \hat{H}^{(H)}] = 0$$

$\Rightarrow \hat{P}_i(t) = \hat{P}_i(0) \Rightarrow$ conservation momentum.

since $[\hat{x}_i^{(H)}, \hat{H}^{(H)}] = [\hat{x}_i, \frac{\hat{P}_j^2}{2m}]$

$$\boxed{[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\cdot\hat{C} + \hat{B}[\hat{A}, \hat{C}]}$$

$$= \underbrace{[\hat{x}_i, \hat{P}_j]}_{i\hbar\delta_{ij}\mathbb{1}}\frac{\hat{P}_j}{2m} + \frac{\hat{P}_j}{2m}[\hat{x}_i, \hat{P}_j]$$

$$= i\hbar\frac{\hat{P}_i}{m}$$

$$\Rightarrow \boxed{\frac{d}{dt}\hat{x}_i(t) = \frac{1}{i\hbar}[\hat{x}_i^{(H)}, \hat{H}^{(H)}] = \frac{1}{i\hbar}\cdot i\hbar\frac{\hat{P}_i}{m} = \frac{\hat{P}_i}{m}}$$

$$\int_0^t d\hat{x}_i(t') = \int_0^t dt' \frac{\hat{P}_i^{(H)}(t)}{m}$$

$$\Rightarrow \quad \hat{x}_i(t) = \hat{x}_i(0) + \frac{\hat{P}_i^{(H)}(t)}{m}\cdot t$$

Further thoughts:

$[\hat{x}_i^{(H)}(0), \hat{x}_j^{(H)}(0)] = 0$, How about $[\hat{x}_i^{(H)}(t), \hat{x}_i^{(H)}(0)]$

$$[\hat{x}_i^{(H)}(t), \hat{x}_i^{(H)}(0)] = [(\hat{x}_i(0) + \frac{\hat{p}_i^{(H)}}{m}t), \hat{x}_i^{(H)}(0)]$$

$$= \frac{\hat{p}_i^{(H)}(0)}{m}t \cdot \hat{x}_i^{(H)}(0) - \hat{x}_i^{(H)}(0) \cdot \frac{\hat{p}_i^{(H)}(0)}{m}t$$

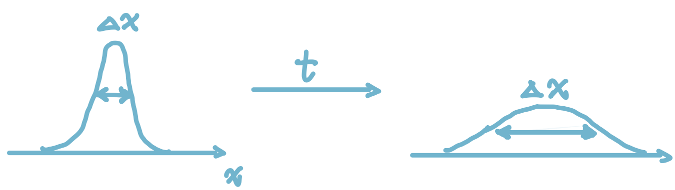
$$= [\hat{p}_i(0), \hat{x}_i(0)] \cdot \frac{t}{m}$$

$$= -i\hbar \cdot \frac{t}{m}$$

recall uncertainty: $\boxed{\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2}$ *

$$\Rightarrow \langle(\Delta\hat{x}_i(t))^2\rangle\langle(\Delta\hat{x}_i(0))^2\rangle$$

$$\geq \frac{1}{4}|\langle -i\hbar\frac{t}{m}\rangle|^2 = \frac{1}{4}\frac{\hbar^2 t^2}{m^2}$$



wave packet of particle is spreading

i.e., the region of finding the particle is larger & larger

in time evolution.

(Also wave mechanics in QM on Group velocity & phase velocity)

Now, add a potential : $V(\hat{x})$

$$V(\hat{x}) \rightarrow (\hat{x}_i, \hat{x}_j, \hat{x}_k)$$

$$\Rightarrow \quad \hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$$

Now $\frac{d\hat{P}_i}{dt} = \frac{1}{i\hbar}[\hat{P}_i, \hat{H}] = \frac{1}{i\hbar}\underbrace{[\hat{P}_i, V(\hat{x})]}_{?}$

if we have $\hat{P}_i$, $g(\hat{x})$, $[\hat{P}_i, g(\hat{x})] = ?$

since $g(\hat{x}) = \sum_{n=0} \frac{g^{(n)}(0)}{n!}(\hat{x})^n$

& $\boxed{[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\cdot\hat{C} + \hat{B}[\hat{A}, \hat{C}]}$

$$\Rightarrow [\hat{P}_i, g(\hat{x})] = -i\hbar \frac{\partial g}{\partial \hat{x}_i}$$

?

$$[\hat{x}_i, f(\hat{P})] = i\hbar \frac{\partial f}{\partial \hat{P}_i}$$

$$\Rightarrow \frac{d\hat{P}_i}{dt} = \frac{1}{i\hbar}\left[-i\hbar \frac{\partial V(\hat{x})}{\partial \hat{x}_i}\right] = -\frac{\partial V(\hat{x})}{\partial \hat{x}_i}$$

$$\frac{d\hat{x}_i}{dt} = \frac{1}{i\hbar}[\hat{x}_i^{(H)}, \hat{H}]$$

$$= \frac{1}{i\hbar}\left[\hat{x}_i^{(H)}, \left(\frac{\hat{P}^2}{2m} + V(\hat{x})\right)\right] \qquad \hat{x} \rightarrow \hat{x}_i, \hat{x}_j, \hat{x}_k$$

$$= \frac{1}{i\hbar}\left[\hat{x}_i^{(H)}, \frac{\hat{P}^2}{2m}\right] = \frac{\hat{P}_i}{m}$$